12.4 Transform of Periodic Sequences

Here we study the DTFT of periodic sequences. We'll start by looking at the Fourier Series expansion, analogous to what we did in continuous time. Then we will derive the same result using a different approach that will lead us into the Discrete Fourier Transform for finite length sequences.

Recall that for **continuous time periodic signals**, we found the Fourier transform by first doing a Fourier series expansion

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \text{synthesis equation} \tag{1}$$

$$a_k = \frac{1}{T}\int_{<T>} x(t)e^{-jk\omega_0 t}dt \qquad \text{analysis equation} \tag{2}$$

then using the fact that a complex exponential in time transforms to an impulse in the frequency domain

$$e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega - \omega_0)$$

and linearity of the Fourier transform, we get that the CTFT of a periodic signal is made up of harmonically-related impulses with area $2\pi a_k$

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$$

Discrete-time periodic signals can also be described by a Fourier Series expansion:

$$x[n] = \sum_{k\in<N>} a_k e^{jk\Omega_0 n} \qquad \text{synthesis equation} \tag{3}$$

$$a_k = \frac{1}{N}\sum_{k\in<N>} x[n]e^{-jk\Omega_0 n} \qquad \text{analysis equation} \tag{4}$$

As one would expect, the integral in time goes to a sum. However, there is one more key difference: *the sum in the synthesis equation is finite!* (over an interval the length of a one period). Since $e^{jk\Omega_0 n} = e^{j(k+N)\Omega_0 n}$, the a_k's are periodic with period N and only N terms are needed in the sum.

So, we have expressed periodic $x[n]$ as a finite sum of complex exponentials with discrete frequencies $k\Omega_0 = \frac{2\pi k}{N}$.

The next step is to find the DTFT of $e^{j\Omega_0 n}$. Since this function is not absolutely summable, we need to allow impulses in order for the DTFT to exist. Hoping that the discrete-time case behaves like continuous time, we might guess ...

$$\begin{aligned} CT: e^{j\omega_0 t} &\longleftrightarrow 2\pi\delta(\omega - \omega_0) && (5) \\ DT: e^{j\Omega_0 n} &\longleftrightarrow 2\pi\delta(\Omega - \Omega_0) \quad ???? && (6) \end{aligned}$$

But this can't be right because the DTFT must be periodic! So, instead let's guess:

$$e^{j\Omega_0 n} \longleftrightarrow 2\pi \sum_{l=-\infty}^{\infty} \delta(\Omega - \Omega_0 + 2\pi l)$$

Then we can verify that this works with our inverse DTFT equation

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{<2\pi>} X(\Omega)e^{j\Omega n} && (7) \\ &= \frac{1}{2\pi}\int_{\Omega_0-\pi}^{\Omega_0+\pi} 2\pi\delta(\Omega - \Omega_0)e^{j\Omega n} && (8) \\ &= e^{j\Omega_0 n} && (9) \end{aligned}$$

taking a 2π interval that contains Ω_0 and using the sifting property of the unit impulse function $\delta(\cdot)$.

Putting together this result with the Fourier Series result, as in continuous time, we get

$$\begin{aligned} X(\Omega) &= 2\pi \sum_{k \in <N>} \sum_{l=-\infty}^{\infty} a_k \delta(\Omega - k\Omega_0 + 2\pi l) && (10) \\ &= 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\Omega - k\Omega_0) && (11) \end{aligned}$$

by periodicity of the $a_k's$.

Noting that $\Omega_0 = \frac{2\pi}{N}$, this gives us Formula 12 of Table 12.1 in the textbook (with typo fixed):

$$x[n] \text{ periodic with period } N \leftrightarrow 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\Omega - \frac{2\pi k}{N})$$

where

$$\begin{aligned} a_k &= \frac{1}{N} \sum_{n=<N>} x[n] e^{-j2\pi nk/N} \\ &= \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} x[n] e^{-j2\pi nk/N} \end{aligned}$$

are Fourier Series coefficients (you sum up over one period of the signal).

$\boxed{\text{Ex.}}$ Fourier Series analysis by inspection. Find and sketch a_k for

$$x[n] = 2 + 2\cos(\frac{\pi}{2}n) + \cos(\frac{\pi}{3}n)$$

$\boxed{\text{Ex.}}$ Find the DTFT of the discrete-time impulse train

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]$$

using the formula for the DTFT of a periodic DT signal and the general method for finding the Fourier Series coefficients.

We see that:

$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN] \leftrightarrow \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\Omega - \frac{2\pi k}{N}) = P(\Omega)$$

Now let's derive this result taking a different approach that will lead to different insights.

Notation: $x[n]$ is a periodic signal with period N. Let $x_0[n]$ be the part of $x[n]$ that is repeated, i.e.

$$x_0[n] = \begin{cases} x[n], & 0 \leq n \leq N-1 \\ 0, & \text{otherwise.} \end{cases}$$

We can take the DTFT of $x_0[n]$:

$$X_0(\Omega) = \sum_{n=-\infty}^{\infty} x_0[n] e^{-jn\Omega} = \sum_{n=0}^{N-1} x_0[n] e^{-jn\Omega}$$

Now, we can also write $x[n]$ as an infinite sum of the function $x_0[n]$ shifted N units at a time:

$$x[n] = \sum_{k=-\infty}^{\infty} x_0[n-kN] = \sum_{k=-\infty}^{\infty} x_0[n] * \delta[n-kN] = x_0[n] * \sum_{k=-\infty}^{\infty} \delta[n-kN]$$

We get from the convolution property that its DTFT $X(\Omega)$ is:

$$x[n] = x_0[n] * p[n] \longleftrightarrow X_0(\Omega) P(\Omega) = X(\Omega)$$

then using the DTFT of the impulse train that we just found

$$\begin{aligned} X(\Omega) &= X_0(\Omega) \left(\frac{2\pi}{N} \sum_k \delta(\Omega - \frac{2\pi k}{N}) \right) && (12) \\ &= \frac{2\pi}{N} \sum_k X_0(\frac{2\pi k}{N}) \delta(\Omega - \frac{2\pi k}{N}) && (13) \end{aligned}$$

by the property of multiplication by a discrete-time impulse.

$\boxed{\text{Ex.}}$ Examine $X_0(\frac{2\pi k}{N})$. How many distinct values does it have?

The inverse DTFT formula is:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\Omega)e^{j\Omega n}d\Omega = \frac{1}{2\pi}\int_0^{2\pi}[\frac{2\pi}{N}\sum_{k=-\infty}^{\infty} X_0(\frac{2\pi k}{N})\delta(\Omega - \frac{2\pi k}{N})]e^{j\Omega n}d\Omega$$

$$= \frac{1}{N}\sum_{k=-\infty}^{\infty} X_0(\frac{2\pi k}{N})\int_0^{2\pi} \delta(\Omega - \frac{2\pi k}{N})e^{j\Omega n}d\Omega = \frac{1}{N}\sum_{k=0}^{N-1} X_0(\frac{2\pi k}{N})e^{\frac{j2\pi kn}{N}}$$

by the sifting property and because only the impulses for k between 0 and $N-1$ occur in the range from 0 to 2π.

Therefore, if we compare to the Fourier Series formulation on page 105, we get that

$$a_k = \frac{1}{N}X_0(\frac{2\pi k}{N})$$